

Inhomogeneous Ising Model on a Multiconnected Network

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We first generalize the inhomogeneous external field Ising model on a ring to include inhomogeneous couplings. We then further generalize the one-dimensional periodic lattice to the simplest multiconnected networks. The fundamental idea and techniques developed here may be also applicable to other problems where topological collective (nonlocal) modes are many fewer in number than total degrees of freedom.

KEY WORDS: Inverse solution; inhomogeneous Ising network; collective modes; topological invariants.

1. INTRODUCTION

The free energy functional format has become increasingly the instrument of choice for the analysis of nonuniform systems in thermal equilibrium. Its routine use in the classical statistical mechanics of fluids⁽¹⁾ was joined long ago by analogous procedures for fermion fluid ground states,⁽²⁾ and more recently by complete solution of model classical lattice gases.⁽³⁾ The models in question were initially one-dimensional—or Bethe lattice generalizations—and produced the striking result that, in this context, nearest neighbor interaction gave rise to nearest direct correlations and hence to nearest neighbor free energy representations. Put in another way, the inverse formulation, in which one asks for the applied potential associated with a given magnetization profile, maintains the nearest neighbor character of the interaction in the presence of arbitrary non-uniformity. This correspondence is broken, although the precise mechanism

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is not known, in the presence of next nearest neighbor interaction.⁽⁵⁾ A recent result is that it is also broken for a one-dimensional lattice with the topology of a circle.⁽⁶⁾ In fact, it seems clear that any interaction network which is not simply connected will have nonlocal modes associated with any circuit of interactions, and that these will appear explicitly in magnetization profile or free energy.

In the recent work alluded to above, it was also shown that the fashion in which global collective variables enter may be conceptually very simple. To be precise, the ring model could be regarded as existing on the extended space of local magnetization and global collective variable, with external potential conjugate to site magnetization at fixed collective amplitude, whereas the conjugate to the collective amplitude was zero, suggesting a dynamical infinite mass. In the present paper, we will examine the extent to which the suggested concepts apply to more complicated networks, supplying hopefully a topological background to the analysis of complex interactions networks. For this purpose, we devote our attention to the next level of complication, in which the circuit contains multibranch nodes and hence several connected loops.

In summary, we first generalize the previous one-loop Ising network analysis to the case of inhomogeneous coupling, not merely to decorate a simple framework, but rather to serve as a tool for the ensuing computations. We then proceed to the two-loop case, employing the artifice of regarding a whole loop as a superbond. The result is again an extended space representation, now requiring a collective parameter for each branch and one for the pair of nodes. Their global nature is reinforced by the observation that the partition function can be expressed in terms of the collective parameters alone. There follows a general proof that any system with the structure now uncovered can be expressed in terms of a free energy on the extended space of site magnetization and collective amplitudes. We then return briefly to the two-node multichannel case and quote the required generalization.

2. GENERALIZATION OF THE ONE-LOOP CASE

We begin by generalizing our one-loop (one-dimensional ring) result to the case in which nearest-neighboring coupling is also inhomogeneous. Consider an Ising spin system $\{\sigma_x\}$, $x=1, \dots, N$, on a ring (Fig. 1); the partition function is

$$Z = \sum_{\{\sigma_x\}} \prod_{x=1}^N w_x(\sigma_x) e_x(\sigma_x, \sigma_{x+1}) \quad (1)$$

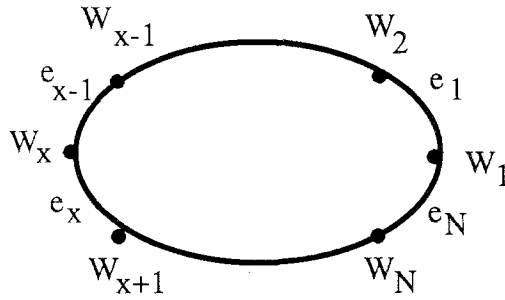


Fig. 1.

where $w_x(\sigma) = (v_x)^\sigma = \exp(h_x \sigma)$, $e_x(\sigma, \sigma') = \exp(J_x \sigma \sigma')$, and $\sigma_{x+1} \equiv \sigma_1$, or, in obvious matrix notation,

$$Z = \text{Tr} \prod_{x=1}^N \mathbf{w}_x \mathbf{e}_x = \text{Tr} \mathbf{w}_1 \mathbf{e}_1 \cdots \mathbf{w}_N \mathbf{e}_N \tag{2}$$

Here, all energies are in units of kT .

By introducing a normalized site-excised function, the matrix

$$\zeta_x = \mathbf{e}_x \mathbf{w}_{x+1} \mathbf{e}_{x+1} \mathbf{w}_{x+2} \cdots \mathbf{w}_{x-2} \mathbf{e}_{x-2} \mathbf{w}_{x-1} \mathbf{e}_{x-1} \mathbf{Z}^{-1} \tag{3}$$

we have the following basic recursive relation:

$$\zeta_x \mathbf{w}_x \mathbf{e}_x = \mathbf{e}_x \mathbf{w}_{x+1} \zeta_{x+1} \tag{4}$$

If we define

$$\bar{\eta}_x = \mathbf{w}_x \zeta_x \quad \text{and} \quad \eta_x = \zeta_x \mathbf{w}_x \tag{5}$$

or

$$\bar{\eta}_x = \mathbf{w}_x \eta_x \mathbf{w}_x^{-1} \tag{6}$$

then (4) is equivalent to

$$\bar{\eta}_{x+1} = \mathbf{e}_x^{-1} \eta_x \mathbf{e}_x \tag{7}$$

From these relations, we see that

$$\begin{aligned} 1 &= \text{Tr} \eta_x = \text{Tr} \bar{\eta}_x \\ K &= \text{Det} \zeta_x = \text{Det} \eta_x = \text{Det} \bar{\eta}_x \\ m\hat{x} &= \text{Tr} \sigma \eta_x = \text{Tr} \sigma \bar{\eta}_x \\ m_{x+1} &= \text{Tr} \sigma \bar{\eta}_{x+1} = \text{Tr} \sigma \mathbf{e}_x^{-1} \eta_x \mathbf{e}_x \\ m_{x-1} &= \text{Tr} \sigma \eta_{x-1} = \text{Tr} \sigma \mathbf{e}_{x-1} \bar{\eta}_x \mathbf{e}_{x-1}^{-1} \end{aligned} \tag{8}$$

where $\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the Pauli matrix, m_x is the mean magnetization at site x , and K is a site-independent function, i.e., the global collective mode (K is related to the variable C in ref. 6 by $C = 1 - 4K$). Actually, using the definition of η_x , we can explicitly calculate the determinant in (7) (note that $\text{Det } \mathbf{w}_x = 1$)

$$K = \text{Det} \left(Z^{-1} \prod_{x=1}^N \mathbf{w}_x \mathbf{e}_x \right) = Z^{-2} \prod_{x=1}^N \text{Det}(\mathbf{e}_x) \quad (9)$$

We might have expected the ubiquity of such a K if we had rewritten (4) in the form

$$\mathbf{Q}_x^{-1} \bar{\eta}_x \mathbf{Q}_x = \bar{\eta}_{x+1} \quad (10)$$

where $\mathbf{Q}_x \equiv \mathbf{w}_x \mathbf{e}_x$ (of course, one may use the dual equation for $\bar{\eta}$). Therefore, we see that the translation of $\bar{\eta}(\eta)$ is equivalent to a local similarity transformation which preserves its eigenvalues λ_1, λ_2 (they are invariant under translation!). From (7), $K = \lambda_1 \lambda_2$, with the conservation law

$$\text{Tr } \eta_x = \lambda_1 + \lambda_2 = 1$$

It is obvious that the eigenvalues are the two roots of the characteristic equation

$$\lambda^2 - \lambda + K = 0$$

In terms of C , $\lambda_{1,2} = (1 \pm \sqrt{C})/2$.

From (9), we find

$$\begin{aligned} \eta_x &= f(m_x, m_{x+1}; \mathbf{e}_x^{-1}, K) \\ \bar{\eta}_x &= f(m_x, m_{x-1}; \mathbf{e}_{x-1}, K) \end{aligned} \quad (11)$$

or, explicitly,

$$\begin{aligned} \eta_x(\pm 1, \pm 1) &= \bar{\eta}_x(\pm 1, \pm 1) = (1 \pm m_x)/2 \\ \eta_x(-1, 1) &= \frac{a_x + [a_x^2 + (1 - m_x^2 - 4K)]^{1/2}}{2} \\ \bar{\eta}_x(-1, 1) &= \frac{-b_x + [b_x^2 + (1 - m_x^2 - 4K)]^{1/2}}{2} \end{aligned} \quad (12)$$

where

$$\begin{aligned} a_x &\equiv \cosh(2J_x) m_x - \sinh(2J_x) m_{x+1} \\ b_x &\equiv \cosh(2J_{x-1}) m_x - \sinh(2J_{x-1}) m_{x-1} \end{aligned} \quad (13)$$

To get the profile equation, we simply use the off-diagonal element of (6) and find

$$v_x^2 = \frac{\bar{\eta}_x(1, -1)}{\eta_x(1, -1)} = \frac{\eta_x(-1, 1)}{\bar{\eta}_x(-1, 1)} = \frac{\{a_x + [a_x^2 + (1 - m_x^2 - 4K)]^{1/2}\} \{-b_x + [b_x^2 + (1 - m_x^2 - 4K)]^{1/2}\}}{1 - m_x^2 - 4K} \quad (14)$$

which immediately generalizes our previous result to inhomogeneous interactions. But the key result is unchanged: the profile equation in inverse form (v_x in terms of the m_x) maintains the nearest neighbor character of the interaction, with the global amplitude K inserted by the ring topology serving as a site-independent parameter to be determined self-consistently (see Section 4).

3. SOLUTION FOR THE TWO-LOOP CASE

Now let us take a generic two-loop lattice, which consists of three chains E^1, E^2, E^3 and two nodes A, B as shown in Fig. 2. We define new composite variables, with obvious notation,

$$E^\alpha = e_A^\alpha w_{A+1}^\alpha e_{A+1}^\alpha \cdots w_{A+N_\alpha}^\alpha e_{A+N_\alpha}^\alpha \quad (15)$$

$$\bar{Q}(\sigma, \sigma') = w_A(\sigma) \prod_{\alpha=1}^3 E^\alpha(\sigma, \sigma') w_B(\sigma') \quad (16)$$

Hence

$$Z = \sum_{\sigma, \sigma'} \bar{Q}(\sigma, \sigma') \quad (17)$$

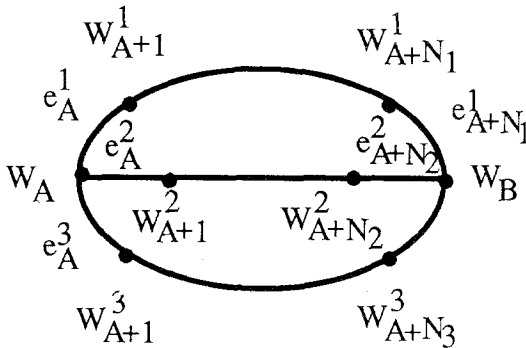


Fig. 2.

and the two-point correlation function between A and B is given by

$$Q(\sigma, \sigma') = \bar{Q}(\sigma, \sigma')/Z \tag{18}$$

Finding v_x for $x \neq A, B$ is easy, since we may treat the system as a ring with a superbond. For instance, supposing x to be on the first chain (Fig. 3), we can define a superbond $E^{23}(\sigma, \sigma') \equiv E^2(\sigma, \sigma')E^3(\sigma, \sigma')$ between A and B . The solution will still be given by (14) with K depending upon which chain the concerned site lies:

$$K^\alpha = \text{Det}(\mathbf{E}^\alpha) \text{Det}(\mathbf{E}^{\beta\gamma})/Z^2 \quad \text{with } \alpha \neq \beta \neq \gamma \tag{19}$$

If x is one of the nodes, the situation is much more complicated. Since we are seeking a local profile which may consist of one more (in addition to the K^α) collective variable characterizing the node's structure, by symmetry, we only have to consider the case when $x = A$.

With the obvious generalization η_x^α for each channel, we have the following relation (Fig. 4):

$$\bar{\eta}_A^\alpha = \mathbf{w}_A \mathbf{E}^{\beta\gamma} \mathbf{w}_B (\mathbf{E}^\alpha)^T$$

or

$$\bar{\eta}_A^\alpha(\sigma, -\sigma) = \sum_{\sigma' = \pm\sigma} \frac{Q(\sigma, \sigma')}{E^\alpha(\sigma, \sigma')} E^\alpha(-\sigma, \sigma') \tag{20}$$

which yields [with $r^\alpha \equiv \bar{\eta}_A^\alpha(1, -1) \bar{\eta}_A^\alpha(-1, 1) = (1 - m_A^2)/4 - K^\alpha$]

$$\begin{aligned} & \frac{E^\alpha(\sigma, \sigma)}{E^\alpha(-\sigma, \sigma)} \\ &= \frac{r_A^\alpha + Q(\sigma, \sigma) Q(-\sigma, \sigma) - Q(-\sigma, -\sigma) Q(\sigma, -\sigma)}{2\bar{\eta}_A^\alpha(\sigma, -\sigma) Q(-\sigma, \sigma)} \\ &+ \frac{\{[r_A^\alpha + Q(\sigma, \sigma) Q(-\sigma, \sigma) - Q(-\sigma, -\sigma) Q(\sigma, -\sigma)]^2 - 4Q(\sigma, \sigma) Q(-\sigma, \sigma) r_A^\alpha\}^{1/2}}{2\bar{\eta}_A^\alpha(\sigma, -\sigma) Q(-\sigma, \sigma)} \end{aligned} \tag{21}$$

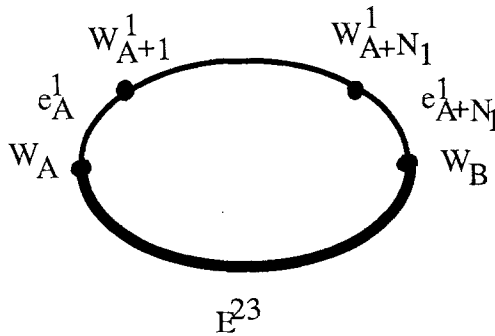


Fig. 3.

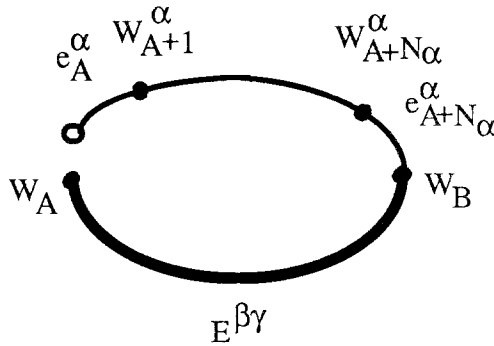


Fig. 4.

where $\bar{\eta}_A^\alpha$ can be expressed in terms of m_A , m_{A+1}^α , and K^α according to (11).

Since Q represents the correlation between σ_A and σ_B , it is intrinsically a nonlocal function. To find a local expression for it (using either the magnetizations about A or about B , but not about both), we have to introduce one additional collective mode parameter which is symmetric with respect to A and B . To do so, we first notice that

$$\frac{Q(1, \sigma)}{Q(-1, \sigma)} = \frac{W_A(1)}{W_A(-1)} \prod_\alpha \frac{E^\alpha(1, \sigma)}{E^\alpha(-1, \sigma)} \tag{22}$$

which implies

$$\frac{Q(1, 1) Q(-1, -1)}{Q(-1, 1) Q(1, -1)} = \prod_\alpha \frac{E^\alpha(1, 1) E^\alpha(-1, -1)}{E^\alpha(-1, 1) E^\alpha(1, -1)} \tag{23}$$

If we define $\theta(Q)$, $\theta_\alpha(Q)$ by

$$e^\theta \equiv \sqrt{\frac{Q(-1, 1) Q(1, -1)}{Q(1, 1) Q(-1, -1)}} \tag{24}$$

$$e^{\theta_\alpha} \equiv \sqrt{\frac{Q(-1, 1) Q(1, -1) E^\alpha(1, 1) E^\alpha(-1, -1)}{Q(1, 1) Q(-1, -1) E^\alpha(-1, 1) E^\alpha(1, -1)}} \tag{25}$$

then the fourth collective mode can be defined as

$$K^4 = \frac{\cosh(\theta)}{\prod_\alpha \cosh(\theta_\alpha)} - 1 \tag{26}$$

For Q , we observe that since

$$Q(\pm 1, \pm 1) + Q(\pm 1, \mp 1) = \frac{1 \pm m_A}{2} \tag{27}$$

together with (23) and (26), Q can be determined as a matrix function of m_A, m_{A+1}^α for fixed K^α and K^4 . The solution, which involves a tedious fourth order algebraic equation, is shown in the Appendix.

Finally, from (22), one can easily see that

$$v_A^2 = \frac{Q(1, 1)}{Q(-1, 1)} \prod_\alpha \frac{E^\alpha(-1, 1)}{E^\alpha(1, 1)} \tag{28}$$

is also a local function of the magnetizations for fixed parameters. And by complete analogy, v_B may also be obtained. In conclusion, h_x depends only on m_x and $m_{x'}$ (where x' is a nearest-neighbor site of x). For x on chain α , h_x has only K^α as its parameter (couplings J_x are considered as constants); for $x = A, B$, it has all four parameters K^1, K^2, K^3 and K^4 .

4. THE NATURE OF COLLECTIVE MODES AND THEIR RELATION TO THE PARTITION FUNCTION

From the definition of the K^α in (19), it is clear that if we break any bond in the α channel, i.e., $J_x^\alpha = 0$, then e_x^α and hence E^α will become singular (its determinant will become zero), so that $K^\alpha = 0$ and $K^\beta = K^\gamma$. Furthermore, from (25) and (24), we see that $\theta = \theta_\alpha$ and $\theta_\beta = \theta_\gamma = 0$. Therefore, according to (26), $K^4 = 0$.

As another special case, if we have uniform magnetization on all the chains except that m_A, m_B may be arbitrary, then $K^\alpha = K^\beta = K^\gamma$; hence $2\theta/3 = \theta_\alpha = \theta_\beta = \theta_\gamma$, and $K^4 = \cosh(\theta)/\cosh^3(2\theta/3) - 1$.

In order to explore the relation between the collective modes and the partition function, let us rewrite (19) as

$$d_\beta s_\gamma + d_\gamma s_\beta = \frac{K^\alpha}{4d_\alpha} Z^2 \tag{29}$$

where

$$d_\alpha \equiv (\text{Det } E^\alpha)/2 = \frac{1}{2} \prod_{i=0}^{N_z} \text{Det } e_{A+i}^\alpha$$

and

$$s_\alpha \equiv \frac{E^\alpha(1, 1) E^\alpha(-1, -1) + E^\alpha(1, -1) E^\alpha(-1, 1)}{2}$$

Solving this set of equations, we have

$$s_\alpha = \frac{K^\beta + K^\gamma - K^\alpha}{8d_\beta d_\gamma} Z^2 \tag{30}$$

On the other hand, from (23)–(25),

$$\theta = -\frac{1}{2} \ln \prod_\alpha \frac{s_\alpha + d_\alpha}{s_\alpha - d_\alpha} = -\sum_\alpha \tanh^{-1} \left(\frac{d_\alpha}{s_\alpha} \right) \tag{31}$$

By substitution of s_α and using the definition (26), we obtain

$$(Z)^4 = \frac{\prod_\alpha d_\alpha^2}{K^4} \sum_{\text{permutations}} \frac{4}{(K^2 + K^3 - K^1)(K^3 + K^1 + K^2)} \tag{32}$$

As in the case of a ring, the partition function can be entirely expressed in terms of the collective modes alone. Thus, the amplitudes K^α control both the profile, via (21), (27), and (28), and the thermodynamics, via (32). Our next task is to determine the K^α .

5. LOCAL FREE ENERGY DENSITY FUNCTIONAL IN THE EXTENDED SPACE

For a lattice on a simple ring, with a single collective amplitude K , we were able to show previously that a free energy could be defined on the extended space (\mathbf{m}, K) which served as generating function for the applied potential field conjugate to \mathbf{m} , determined K by the condition that its conjugate vanish, and reduced to the physical free energy function on \mathbf{m} upon insertion of $K = K(\mathbf{m})$. In the present situation, where we have the set of amplitudes $\mathbf{K} = (K^1, K^2, K^3, K^4)$, and the corresponding extended space (\mathbf{m}, \mathbf{K}) , a fully analogous result applies.

The argument is quite general. Suppose that there are d collective variables K^1, \dots, K^d . Given sites x and y , if the system is large enough, we can choose sites z_1, \dots, z_d such that $|z_\alpha - x| > 1$, $|z_\alpha - y| > 1$ for each α . Now let us examine the chain rule relation (Einstein summation convention used throughout)

$$\frac{\partial h_x}{\partial m_y} = \left. \frac{\partial h_x}{\partial m_y} \right|_K + \left. \frac{\partial h_x}{\partial K^\nu} \right|_m \frac{\partial K^\nu}{\partial m_y} \tag{33}$$

where x runs over the sites z_α . We have seen that h_x is local in the $\{m_y\}$

at fixed \mathbf{K} , containing at most nearest neighbor terms. Thus, with the above choice of sites, the first term on the right will vanish:

$$\frac{\partial h_{z_\alpha}}{\partial m_y} = \frac{\partial h_{z_\alpha}}{\partial K^v} \bigg|_m \frac{\partial K^v}{\partial m_y} \quad (34)$$

Since the h_{z_α} are independent, $\partial h_{z_\alpha}/\partial K^v|_m$ is invertible, and by integrability $\partial h_{z_\alpha}/\partial m_y = \partial h_y/\partial m_{z_\alpha}$, so that

$$\frac{\partial K^v}{\partial m_y} = \frac{\partial K^v}{\partial h_{z_\alpha}} \bigg|_m \frac{\partial h_y}{\partial m_{z_\alpha}} \quad (35)$$

or, according to (33),

$$\frac{\partial K^v}{\partial m_y} = \frac{\partial K^v}{\partial h_{z_\alpha}} \bigg|_m \frac{\partial h_y}{\partial K^\mu} \bigg|_m \frac{\partial K^\mu}{\partial m_{z_\alpha}} \quad (36)$$

Hence (33) becomes

$$\begin{aligned} \frac{\partial h_x}{\partial m_y} &= \frac{\partial h_x}{\partial m_y} \bigg|_K + \frac{\partial h_x}{\partial K^v} \bigg|_m \frac{\partial K^v}{\partial h_{z_\alpha}} \bigg|_m \frac{\partial h_y}{\partial K^\mu} \bigg|_m \frac{\partial K^\mu}{\partial m_{z_\alpha}} \\ &= \frac{\partial h_x}{\partial m_y} \bigg|_K + \frac{\partial h_x}{\partial K^v} \bigg|_m \frac{\partial h_y}{\partial K^\mu} \bigg|_m \frac{\partial K^v}{\partial h_{z_\beta}} \bigg|_m \frac{\partial h_{z_\alpha}}{\partial m_{z_\beta}} \frac{\partial K^\mu}{\partial h_{z_\alpha}} \end{aligned} \quad (37)$$

The lhs of (37) and the second term on the rhs of (37) are symmetric in x and y , and so we conclude that $\partial h_x/\partial m_y|_K$ is symmetric as well:

$$\frac{\partial h_x}{\partial m_y} \bigg|_K = \frac{\partial h_y}{\partial m_x} \bigg|_K \quad (38)$$

It follows from the integrability condition represented by (38) that there is a free energy on the expanded space such that

$$h_x = \frac{\partial F[\mathbf{m}, \mathbf{K}]}{\partial m_x} \bigg|_K \quad (39)$$

But we also know that there is an $\bar{F}(\mathbf{m})$ for which $h_x = \partial \bar{F}/\partial m_x$. Hence, as in refs. 4 and 6,

$$0 = \partial(\bar{F}(\mathbf{m}) - F[\mathbf{m}, \mathbf{K}])/\partial m_x + \partial F/\partial K^\alpha \partial K^\alpha/\partial m_x$$

which implies that $\bar{F} - F$ and the K^α are functionally related, i.e.,

$$\bar{F}[\mathbf{m}] = F[\mathbf{m}, \mathbf{K}(\mathbf{m})] + W(\mathbf{K}(\mathbf{m})) \quad (40)$$

for some function W . We conclude therefore on defining

$$\bar{F}[\mathbf{m}, \mathbf{K}] = F[\mathbf{m}, \mathbf{K}] + W(\mathbf{K}) \tag{41}$$

that

$$\begin{aligned} \bar{F}[\mathbf{m}, \mathbf{K}(\mathbf{m})] &= \bar{F}[\mathbf{m}] \\ h_x &= \frac{\partial \bar{F}[\mathbf{m}, \mathbf{K}]}{\partial m_x} \\ 0 &= \frac{\partial \bar{F}[\mathbf{m}, \mathbf{K}]}{\partial \mathbf{K}} \Big|_{\mathbf{K}=\mathbf{K}(\mathbf{m})} \end{aligned} \tag{42}$$

From definition, we know that

$$\bar{F}[\mathbf{m}] = \sum_x m_x h_x - \ln Z \tag{43}$$

and

$$\begin{aligned} F[\mathbf{m}, \mathbf{K}] &= \sum_x m_x \int_0^1 d\lambda h_x(\lambda \mathbf{m}, \mathbf{K}) \\ &= \sum_x m_x h_x(\mathbf{m}, \mathbf{K}) - \sum_x m_x \int_0^1 \lambda \frac{\partial h_x(\lambda \mathbf{m}, \mathbf{K})}{\partial \lambda} d\lambda \end{aligned} \tag{44}$$

From (40), we get

$$W(\mathbf{K}(\mathbf{m})) = \sum_x m_x \int_0^1 \lambda \frac{\partial h_x(\lambda \mathbf{m}, \mathbf{K}(\mathbf{m}))}{\partial \lambda} d\lambda - \ln Z(\mathbf{K}(\mathbf{m})) \tag{45}$$

In order to construct $\bar{F}(\mathbf{m}, \mathbf{K})$, we need to know W . On the other hand, according to (45), $W(\mathbf{K})$ can be found if $\mathbf{m}(\mathbf{K})$ is known for any special case with four adjustable parameters. There is no particular virtue to exhibiting the explicit form of $W(\mathbf{K})$ —this has been done in the case of a simple ring—but the procedure is easily described. We may take, for instance, the special piecewise uniform system defined by $m_A = m_B = m^A$, $m_x^\alpha = m^\alpha$. Now \mathbf{K} can be calculated directly by using their definitions (19) and (26) together with (23)–(25), because \mathbf{E}^α depends only on h_x for $x \neq A, B$, which is a simple function of local \mathbf{m} and K^α , (14) [see the paragraph above (19)]. Since Z as a function of \mathbf{K} is also known (32), from these definitions of \mathbf{K} we should then solve for $\mathbf{m}(\mathbf{K})$ and substitute back into (45) to get $W(\mathbf{K})$ explicitly. With this W , the free energy functional in the extended space becomes

$$\bar{F}[\mathbf{m}, \mathbf{K}] = W(\mathbf{K}) + \sum_x m_x \int_0^1 d\lambda \frac{\partial h_x(\lambda \mathbf{m}, \mathbf{K})}{\partial \lambda} \tag{46}$$

It is clear that, since $h_x(\mathbf{m}, \mathbf{K})$ is local, so is \bar{F} .

6. GENERALIZATION TO MANY CHANNELS

Almost all of the previous discussion can be extended to the case where α runs from 1 to n (there are $n-1$ independent loops). The necessary changes are the following:

θ should be defined as

$$e^\theta \equiv \left(\frac{Q(-1, 1) Q(1, -1)}{Q(1, 1) Q(-1, -1)} \right)^{(n-2)/2} \quad (47)$$

Equation (29) should be replaced by

$$\prod_{\beta \neq \alpha} (s_\beta + d_\beta) - \prod_{\beta \neq \alpha} (s_\beta - d_\beta) = \frac{K^\alpha}{2d_\alpha} Z^2 \quad (48)$$

The solution s_α should be substituted into the expression for the $(n+1)$ th collective variable

$$K^{n+1} = \sum_{\text{permutations}} \left(\prod_{\alpha=1}^2 \tanh \theta_\alpha + \prod_{\alpha=1}^4 \tanh \theta_\alpha + \cdots + \prod_{\alpha=1}^{2[n/2]} \tanh \theta_\alpha \right) \quad (49)$$

where $\theta_\alpha = (2-n) \tanh^{-1}(d_\alpha/s_\alpha)$. This is the equation which determines $Z(\mathbf{K})$ instead of (32).

7. CONCLUSION

The dynamical variables of an Ising lattice are the ensemble-averaged site magnetizations. Equilibrium in the one-dimensional lattice with nearest-neighbor coupling is characterized⁽³⁾ by a free energy which likewise couples only nearest neighbors. When the topology is altered by imposing periodic boundary conditions, this locality is broken by the appearance of a global mode variable.⁽⁶⁾ We have here examined the topological situation of several circuits with a common pair of nodes, and found that each arm of the network introduces a collective mode, in addition to one emanating from the pair of nodes. The free energy structure is again nearest-neighbor local on the extended magnetization-mode space, with the mode conjugate variables vanishing in equilibrium. This situation appears generic for Ising networks, and we intend to show in a future communication how the techniques of this paper apply, with some alternation, to analysis of a general Ising network.

APPENDIX

In this Appendix, we derive the fourth-order algebraic equation which generates a solution for $Q(m_A, m_{A+1}^\alpha; K^\alpha, K^4)$. We start by defining new variables

$$\begin{aligned} x &= Q(1, 1) Q(-1, 1), & y &= Q(-1, -1) Q(1, -1) \\ z &= Q(-1, 1), & \bar{z} &= Q(1, -1) \end{aligned} \tag{A1}$$

and

$$u = x + y, \quad v = (xy)^{1/2} \tag{A2}$$

With these variables, Eqs. (27) and (24) become

$$\begin{aligned} \frac{x}{z} + \bar{z} &= \frac{1 + m_A}{2} \\ \frac{y}{\bar{z}} + z &= \frac{1 - m_A}{2} \\ z\bar{z} &= (xy)^{1/2} e^\theta \end{aligned} \tag{A3}$$

From these, we get

$$2v \cosh \theta + u = (1 - m_A^2)/4 \equiv M_A \tag{A4}$$

On the other hand, (23) means that

$$\theta = \theta_1 + \theta_2 + \theta_3 \tag{A5}$$

Therefore, for K^4 we have (with $t_x \equiv \tanh^2 \theta_x$)

$$K^4 = (t_1 t_2)^{1/2} + (t_2 t_3)^{1/2} + (t_3 t_1)^{1/2} \tag{A6}$$

or equivalently

$$[(K^4)^2 - t_1 t_2 - t_2 t_3 - t_3 t_1]^2 - 4t_1 t_2 t_3 (2K^4 + t_1 + t_2 + t_3) = 0 \tag{A7}$$

Instead of using (22), we can find a more symmetric expression. Taking the product over $\sigma = \pm 1$ on both sides of (20), we get

$$\begin{aligned} r_A^\alpha &= Q(-1, 1) Q(1, -1) \frac{E^\alpha(1, 1) E^\alpha(-1, -1)}{E^\alpha(-1, 1) E^\alpha(1, -1)} \\ &+ Q(-1, -1) Q(1, 1) \frac{E^\alpha(-1, 1) E^\alpha(1, -1)}{E^\alpha(1, 1) E^\alpha(-1, -1)} \\ &+ Q(1, 1) Q(-1, 1) + Q(-1, -1) Q(1, -1) \end{aligned} \tag{A8}$$

or

$$\cosh(\theta_\alpha) = \frac{r_A^\alpha - u}{2v} = \frac{M_A - K^\alpha - u}{2v} \quad (\text{A9})$$

By substituting this into (A4),

$$\cosh(\theta) \equiv (K^4 + 1) \prod_x \left(\frac{M_A - K^\alpha - u}{2v} \right) = \frac{M_A - u}{2v} \quad (\text{A10})$$

(A9) and (A10) give

$$t_\alpha = 1 - (\cosh \theta_\alpha)^{-2} = 1 - (K^4 + 1) \frac{(\bar{u} + K^\beta)(\bar{u} + K^\gamma)}{\bar{u}(\bar{u} + K^\alpha)} \quad (\text{A11})$$

where

$$\bar{u} \equiv u - M_A = u - \frac{1 - m_A^2}{4} \quad (\text{A12})$$

Finally, by substituting (A11) into (A7), we obtain the fourth-order equation for \bar{u} and hence for u . Once u is found, v^2 can also be solved by (A10). The x and y will be given by the two roots of the quadratic equation

$$t^2 - ut + v^2 = 0$$

z and \bar{z} will be given by the first two equations in (A3). Since v_A^2 depends only on x , y , and z/\bar{z} [see (28) and (21)], it can therefore be regarded as solved, too.

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